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Generalized hyperbolic processes autocovariance functions

Abstract. Generalized hyperbolic processes are Levy processes which allow an almost perfect fit to financial data. Autocovariance functions of generalized hyperbolic processes such as the normal inverse Gaussian process, the variance gamma process and the hyperbolic process are deduced at this paper.

Keywords. Generalized hyperbolic process, normal inverse Gaussian process, variance gamma process, autocovariation function.

1. Generalized hyperbolic process moments

Generalized hyperbolic (GH) distributions are defined in [1] through its characteristic function

$$\varphi_X^{GH}(u;\lambda,\alpha,\beta,\delta,\mu) =$$
$$= \exp(iu\mu)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}, u \in \mathbb{R} \quad (1)$$

where $X \sim GH(\lambda,\alpha,\beta,\delta,\mu)$ is a random variable following the generalized hyperbolic distribution,

$K_\lambda(z) = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp\left(-\frac{1}{2}z(u+u^{-1})\right)du$ is the modified Bessel function of the third kind, $z > 0$, $\mu \in \mathbb{R}$

$$\delta \geq 0, |\beta| < \alpha, \text{ if } \lambda > 0,$$
$$\delta > 0, |\beta| < \alpha, \text{ if } \lambda = 0,$$
$$\delta > 0, |\beta| \leq \alpha, \text{ if } \lambda < 0.$$

The generalized hyperbolic distribution depends in five parameters: α determines a shape, β determines a skewness, δ is a scaling parameter, μ determines a location and λ characterizes certain sub-classes.

Definition 1. A stochastic process $H=(H_t)_{t\geq 0}$ with the parameters $\lambda, \alpha, \beta, \delta, \mu$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), P)$ having values in $\mathbb{R}$, such as $H_0 \overset{a.s.}{=} 0$, is called the generalized hyperbolic process if

1) H has independent and stationary increments;

2) H increments $H_{t+s}-H_s$ follow the generalized hyperbolic law with the parameters $\lambda, \alpha, \beta, \delta, \mu$, i.e.

$$H_{t+s}-H_s \overset{D}{=} H_t - H_0 \sim GH(\lambda, \alpha, \beta, \delta t, \mu t),\ s \geq 0, t \geq 0,$$

3) H is stochastically continuous, i.e. for every $t \geq 0$ and $\varepsilon > 0$

$$\lim_{s\to t} P(|H_s - H_t| > \varepsilon) = 0.$$

Theorem 1. The generalized hyperbolic process mean and variance are given by the following equations

$$EH_t = \mu t + \frac{\beta\delta t}{\sqrt{\alpha^2-\beta^2}} \frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}, \tag{2}$$

$$DH_t = \delta^2 t^2 \left(\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta t\sqrt{\alpha^2-\beta^2} K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} + \right.$$
$$\left. + \frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} - \frac{K^2_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K^2_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \right)\right). \tag{3}$$

Proof. According to the infinitely divisible property of the generalized hyperbolic distribution [1] one has

$$\varphi^{GH}_{H_t}(u; \lambda, \alpha, \beta, \delta t, \mu t) = (\varphi^{GH}_{H_1}(u; \lambda, \alpha, \beta, \delta, \mu))^t, \tag{4}$$

where $\varphi^{GH}_{H_1}(u; \lambda, \alpha, \beta, \delta, \mu)$ is defined by (1).

Having used the cumulants method [2] one has

$$EH_t = c_1 = i^{-1} \left. \frac{\partial \ln \varphi^{GH}_{H_t}(u; \lambda, \alpha, \beta, \delta t, \mu t)}{\partial u} \right|_{u=0}.$$

From (1) and (4) one gets

$$EH_t = i^{-1} \frac{\left(\varphi_{H_t}^{GH}(u;\lambda,\alpha,\beta,\delta t,\mu t)\right)_u^{/}}{\varphi_{H_t}^{GH}(u;\lambda,\alpha,\beta,\delta t,\mu t)}\Bigg|_{u=0} =$$

$$= i^{-1} \frac{\left(\left(\varphi_{H_1}^{GH}(u;\lambda,\alpha,\beta,\delta,\mu)\right)^t\right)_u^{/}}{\left(\varphi_{H_1}^{GH}(u;\lambda,\alpha,\beta,\delta,\mu)\right)^t}\Bigg|_{u=0} =$$

$$= i^{-1} \frac{\partial\left(\exp(iu\mu)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^t}{\partial u}\Bigg|_{u=0} =$$

$$= i^{-1} \frac{\left(\alpha^2-\beta^2\right)^{\lambda t/2}}{K_\lambda^t\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\Bigg[\Bigg(\frac{\partial \exp(iu\mu t)}{\partial u}\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2} +$$

$$+ \exp(iu\mu t)\frac{\partial\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2}}{\partial u}\Bigg)\times$$

$$\times K_\lambda^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right) +$$

$$+ \exp(iu\mu t)\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2}\frac{\partial K_\lambda^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{\partial u}\Bigg]\Bigg|_{u=0}.$$

Having differentiated the modified Bessel function of the third kind and taking into account its following properties

$$K_\lambda^{/}(x) = -\frac{\lambda}{x}K_\lambda(x) - K_{\lambda-1}(x),\; K_{\lambda-1}(x) = K_{\lambda+1}(x) - \frac{2\lambda}{x}K_\lambda(x) \quad (5)$$

one gets

$$EH_t = i^{-1}\frac{\left(\alpha^2-\beta^2\right)^{\lambda t/2}}{K_\lambda^t\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left[\left(\frac{i\mu t\exp(iu\mu t)}{\left(\alpha^2-(\beta+iu)^2\right)^{\lambda t/2}}+\right.\right.$$

$$\left.+\frac{i\lambda t(\beta+iu)\exp(iu\mu t)}{\left(\alpha^2-(\beta+iu)^2\right)^{\lambda t/2+1}}\right)\times K_\lambda^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)-$$

$$-K_\lambda^{t-1}\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)\frac{i\delta t(\beta+iu)\exp(iu\mu t)}{\left(\alpha^2-(\beta+iu)^2\right)^{\lambda t/2+1/2}}\times$$

$$\times\left(\frac{\lambda}{\delta\sqrt{\alpha^2-(\beta+iu)^2}}K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)-\right.$$

$$\left.\left.\left.-K_{\lambda+1}^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)\right)\right]\right|_{u=0}=$$

$$=\mu t+\frac{\beta\delta t}{\sqrt{\alpha^2-\beta^2}}\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}.$$

The generalized hyperbolic process variance is found to fit

$$DH_t = c_2(t) = i^{-2}\left.\frac{\partial^2\ln\varphi_{H_t}^{GH}(u;\lambda,\alpha,\beta,\delta t,\mu t)}{\partial u^2}\right|_{u=0}.$$

From (1) and (4) one gets

$$DH_t = -\frac{\partial}{\partial u}\left.\left(\frac{\partial\ln\left(\varphi_{H_1}^{GH}(u;\lambda,\alpha,\beta,\delta,\mu)\right)^t}{\partial u}\right)\right|_{u=0}=$$

$$=-\frac{\partial}{\partial u}\left(-i\frac{\left(\alpha^2-\beta^2\right)^{\lambda t/2}}{K_\lambda^t\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left[\left(\frac{\partial\exp(iu\mu t)}{\partial u}\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2}+\right.\right.\right.$$

$$\left.+\exp(iu\mu t)\frac{\partial\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2}}{\partial u}\right)K_\lambda^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)+$$

$$+\exp(iu\mu t)\left(\alpha^2-(\beta+iu)^2\right)^{-\lambda t/2}\times$$

$$\left.\left.\left.\times\frac{\partial K_\lambda^t\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{\partial u}\right]\right)\right|_{u=0} \qquad (6)$$

Having differentiated (6) and taking into account (5) one gets (3).

2. Generalized hyperbolic processes autocovariance functions

The generalized hyperbolic process $H=(H_t)_{t\ge 0}$ with the parameters $\lambda,\alpha,\beta,\delta,\mu$ can be equated [3, 4]

$$H_t=\mu t+\beta Z_t+W_{Z_t}, \tag{7}$$

where $Z=(Z_t)_{t\ge 0}$ – the generalized inverse Gaussian process with the parameters λ, $a=\delta$, $b=\sqrt{\alpha^2-\beta^2}$, $W=(W_t)_{t\ge 0}$ – the standard Brownian motion, independent of $Z=(Z_t)_{t\ge 0}$.

Theorem 2. The generalized hyperbolic process autocovariance function is given by

$$EH_tH_s=\frac{\delta^2t^2}{2}\left(\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta t\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}+\right.$$

$$\left.+\frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}-\frac{K^2_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K^2_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)\right)+$$

$$+\frac{1}{2}\left(\mu t+\beta\delta t\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^2+$$

$$+\frac{\delta^2s^2}{2}\left(\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta s\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}+\right.$$

$$\left.+\frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}-\frac{K^2_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K^2_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)\right)+$$

$$+\frac{1}{2}\left(\mu s+\beta\delta s\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^2-$$

$$-\frac{\delta^2(t-s)^2}{2}\left(\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\delta(t-s)^2\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}+\right.$$

$$+\frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}-\frac{K_{\lambda+1}^2\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda^2\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)\Bigg)\Bigg)-$$

$$-\frac{1}{2}\left(\mu(t-s)^2+\beta\delta(t-s)^2\frac{K_{\lambda+1}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^2, t\neq s \quad (8)$$

$K_\lambda(z)$ – the modified Bessel function of the third kind, $z>0$.

Proof. Because of (7) the generalized hyperbolic process $H=(H_t)_{t\geq 0}$ mean and variance can be achieved as

$EH_t = E\left(\mu t+\beta Z_t+W_{Z_t}\right)$, $DH_t = EH_t^2-(EH_t)^2$,

where $Z=(Z_t)_{t\geq 0}$ – the generalized inverse Gaussian process, $W=(W_t)_{t\geq 0}$ – the standard Brownian motion.

Hence the generalized hyperbolic process autocovariance function is of the form

$$EH_tH_s=\frac{1}{2}\left[EH_t^2+EH_s^2-E\left(H_t-H_s\right)^2\right], \quad (9)$$

where $EH_t^2 = DH_t+(EH_t)^2$. The generalized hyperbolic process $H=(H_t)_{t\geq 0}$ mean EH_t and variance DH_t are defined by (2), (3).

When $t>s$ one gets

$$EH_t^2=\delta^2t^2\left(\frac{K_{\lambda+1}\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}{\delta t\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}+\right.$$

$$\left.+\frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}-\frac{K_{\lambda+1}^2\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda^2\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}\right)\right)+$$

$$+\left(\mu t+\beta\delta t\frac{K_{\lambda+1}\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta t\sqrt{\alpha^2-\beta^2}\right)}\right)^2, \quad (10)$$

$$EH_s^2 = \delta^2 s^2 \left(\frac{K_{\lambda+1}\left(\delta s\sqrt{\alpha^2-\beta^2}\right)}{\delta s\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta s\sqrt{\alpha^2-\beta^2}\right)} + \right.$$

$$\left. + \frac{\beta^2}{\alpha^2-\beta^2}\left(\frac{K_{\lambda+2}\left(\delta s\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta s\sqrt{\alpha^2-\beta^2}\right)} - \frac{K_{\lambda+1}^2\left(\delta s\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda^2\left(\delta s\sqrt{\alpha^2-\beta^2}\right)} \right) \right) +$$

$$+ \left(\mu s + \beta\delta s \frac{K_{\lambda+1}\left(\delta s\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta s\sqrt{\alpha^2-\beta^2}\right)} \right)^2, \qquad (11)$$

According to the generalized hyperbolic process definition

$$H_t - H_s \overset{d}{=} H_{t-s} - H_0 \overset{d}{=} H_{t-s},$$

then

$$E\left(H_t - H_s\right)^2 = EH_{t-s}^2 = \delta^2 (t-s)^2 \times$$

$$\times \left(\frac{K_{\lambda+1}\left(\delta(t-s)^2\sqrt{\alpha^2-\beta^2}\right)}{\delta(t-s)^2\sqrt{\alpha^2-\beta^2}K_\lambda\left(\delta(t-s)^2\sqrt{\alpha^2-\beta^2}\right)} + \right.$$

$$\left. + \left(\mu(t-s)^2 + \beta\delta(t-s)^2 \frac{K_{\lambda+1}\left(\delta(t-s)^2\sqrt{\alpha^2-\beta^2}\right)}{\left(\alpha^2-\beta^2\right)K_\lambda\left(\delta(t-s)^2\sqrt{\alpha^2-\beta^2}\right)} \right)^2 \right). \qquad (12)$$

Having substituted (10), (11), (12) to (9) one achieves the generalized hyperbolic process autocovariance function when $t > s$.
The case $t < s$ is analogical. Then one has (8) for any $t, s, t \neq s$.

Corollary 1. The hyperbolic process $Y = \left(Y_t\right)_{t\geq 0}$ with the parameters α, β, δ, μ autocovariance function is given by (8) when $\lambda = 1$.

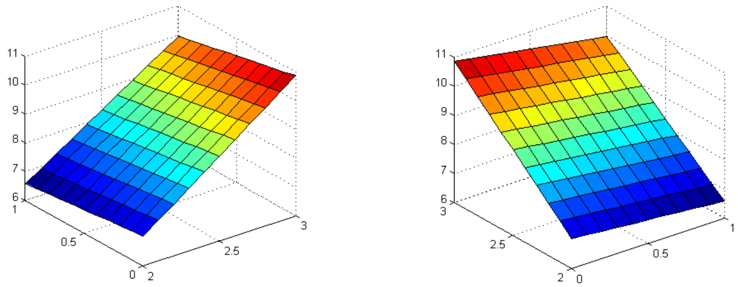
Proof. The proof of the corollary 3 follows from the theorem 2 proof because of the hyperbolic process is a special case of the generalized hyperbolic process when $\lambda = 1$.

Corollary 2. The VG-process $V = \left(V_t\right)_{t\geq 0}$ with the parameters σ, ν, θ autocovariance function is given by

$$EV_tV_s = \min(t,s)(\theta^2\nu + \sigma^2\max(t,s)(\theta+\mu)^2).$$

Proof. The proof of the corollary 2 follows from the theorem 2 proof because of the VG-process is a special case of the generalized hyperbolic process when $\lambda = \sigma^2 / \nu, \delta \to 0$.

The autocovariance function of a VG-process is presented at the figure 1.



a) $\sigma = 1, \nu = 0.5, \theta = 1, \mu = 0, t > s$ b) $\sigma = 1, \nu = 0.5, \theta = 1, \mu = 0, t < s$

Figure 1. The autocovariance function of a VG-process

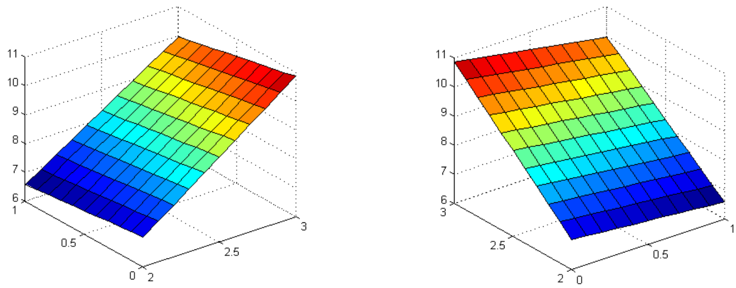
Corollary 3. The normal inverse Gaussion process $N = (N_t)_{t \geq 0}$ with the parameters α, β, δ, μ autocovariance function is given by

$$EN_tN_s =$$

$$= \begin{cases} \dfrac{\alpha^2 \delta t}{(\alpha^2 - \beta^2)^{3/2}} + \dfrac{1}{2}\left(\left(\dfrac{\delta^2 t^2 \beta^2}{\alpha^2 - \beta^2} + \mu t \right)^2 + \left(\dfrac{\delta^2 s^2 \beta^2}{\alpha^2 - \beta^2} + \mu s \right)^2 + \right. \\ \left. + \left(\dfrac{\delta^2 (t-s)^2 \beta^2}{\alpha^2 - \beta^2} + \mu (t-s) \right) \right)^2, t > s, \\ \dfrac{\alpha^2 \delta s}{(\alpha^2 - \beta^2)^{3/2}} + \dfrac{1}{2}\left(\left(\dfrac{\delta^2 t^2 \beta^2}{\alpha^2 - \beta^2} + \mu t \right)^2 + \left(\dfrac{\delta^2 s^2 \beta^2}{\alpha^2 - \beta^2} + \mu s \right)^2 + \right. \\ \left. + \left(\dfrac{\delta^2 (s-t)^2 \beta^2}{\alpha^2 - \beta^2} + \mu (s-t) \right) \right)^2, t < s. \end{cases}$$

Proof. The proof of the corollary 1 follows from the theorem 2 proof because of the normal inverse Gaussion process is a special case of the generalized hyperbolic process when $\lambda = -1/2$.

The autocovariance function of a normal inverse Gaussian process is presented at the figure 2.



a) $\alpha = 1, \beta = 0.5, \delta = 1, \mu = 0, t > s$ b) $\alpha = 1, \beta = 0.5, \delta = 1, \mu = 0, t < s$

Figure 2. The autocovariance function of a normal inverse Gaussian process

The autocovariance functions of the generalized hyperbolic process, normal inverse Gaussion process, VG-process, hyperbolic process are achieved at the paper. The generalized hyperbolic processes autocovariance functions building are carried out in MATLAB® 7.6.0 (R2008a).

References

1. Barndorff-Nielsen O.E., Halgreen C. (1997) *Infinitely divisibility of the hyperbolic and generalized inverse Gaussian distributions* Zeitschrift for Wahrscheinlichkeitstheorie und verwandte Gebiete, Berlin,V. 38, 309–311.
2. Труш Н.Н. (1999) *Асимптотические методы статистического анализа временных рядов*, Минск: БГУ.
3. Ширяев А.Н. (2009) *Финансовые инновации в стохастической экономике,* Экономика и мат. методы, Т. 45 №1. 87-94.
4. Eberlein E. (2001) *Application of generalized hyperbolic Levy motions to finance*, Levy processes: Theory and Applications, Boston: Birkhauser. 319–336.
5. Trusz M., Kuzmina A. V. (2012) *Option pricing by Esscher transforms in the cases of normal inverse Gaussian and variance gamma processes*, Studia Informatica, № 1-2(16). 35-44.